Electronic Companion for Capacity Constraints Across Nests in Assortment Optimization Under the Nested Logit Model

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A Online Appendix: Proof of Corollary 2

We let $(S_1^*, \ldots, S_m^*)$ be an optimal solution to problem (1) so that $z^* = \Pi(S_1^*, \ldots, S_m^*) = \sum_{i\in M} V_i(S_i^*)^{\gamma_i} R_i(S_i^*)/(v_0 + \sum_{i\in M} V_i(S_i^*)^{\gamma_i})$. Focusing on the first and last terms in this chain of equalities and solving for z^*, we obtain $v_0\, z^* = \sum_{i\in M} V_i(S_i^*)^{\gamma_i}(R_i(S_i^*) - z^*)$. Since $(S_1^*, \ldots, S_m^*)$ is a feasible solution to problem (2) when we solve this problem with $z = z^*$, we obtain $f(z^*) \geq \sum_{i\in M} V_i(S_i^*)^{\gamma_i}(R_i(S_i^*) - z^*)$, in which case, using the last equality, we have $f(z^*) \geq v_0\, z^*$. We claim that $\alpha\, \hat{z} \geq z^*$. To get a contradiction, assume that $\alpha\, \hat{z} < z^*$. In this case, we obtain $f(z^*) \geq v_0\, z^* > \alpha\, v_0\, \hat{z} = \alpha\, f^R(\hat{z}) \geq f(\hat{z})$, where the equality follows from the definition of $\hat{z}$. Since $f(\cdot)$ is decreasing, having $f(z^*) \geq f(\hat{z})$ implies that $z^* \leq \hat{z} \leq \alpha\, \hat{z}$, which contradicts the assumption that $\alpha\, \hat{z} < z^*$ and the claim follows. To obtain the desired result, we observe that $\sum_{i\in M} V_i(\hat{S}_i)^{\gamma_i}(\alpha\, \beta\, R_i(\hat{S}_i) - z^*) \geq \sum_{i\in M} V_i(\hat{S}_i)^{\gamma_i}(\alpha\, \beta\, R_i(\hat{S}_i) - \beta\, z^*) \geq \sum_{i\in M} V_i(\hat{S}_i)^{\gamma_i}(\alpha\, \beta\, R_i(\hat{S}_i) - \alpha\, \beta\, \hat{z}) \geq \alpha\, f^R(\hat{z}) = \alpha\, v_0\, \hat{z} \geq v_0\, z^*$, where the first inequality follows from the fact that $\beta \geq 1$, the second inequality is by the fact that $\alpha\, \hat{z} \geq z^*$ and the third inequality follows from the inequality given in the corollary. Focusing on the first and last expressions in the last chain of inequalities and solving for z^*, we obtain $z^* \leq \alpha\, \beta \sum_{i\in M} V_i(\hat{S}_i)^{\gamma_i} R_i(\hat{S}_i)/(v_0 + \sum_{i\in M} V_i(\hat{S}_i)^{\gamma_i}) = \alpha\, \beta\, \Pi(\hat{S}_1, \ldots, \hat{S}_m)$.

B Online Appendix: An Upper Bound

The approach in Section 5 obtains a 4-approximate solution under a space constraint, indicating that this approach never performs arbitrarily badly. However, knowing that a solution provides at least a quarter of the optimal expected revenue may not be thoroughly satisfying from a practical perspective. In this section, we develop a tractable approach for obtaining an upper bound on the optimal expected revenue for an individual instance of problem (1) under a space constraint. By comparing this upper bound on the optimal expected revenue with the expected revenue obtained by a particular assortment, we can get a feel for the optimality gap of the assortment on hand.

To construct an upper bound on the optimal expected revenue in problem (1), for each nest i, we partition the interval $[0, c]$ into K intervals $\{[b_i^{k-1}, b_i^k] : k = 1, \ldots, K\}$, where we have $0 = b_i^0 \leq b_i^1 \leq \ldots \leq b_i^{K-1} \leq b_i^K = c$. Noting that the total preference weight of the products offered in nest i can at most be $\sum_{j\in N} v_{ij}$, we let $\bar{v}_i = \sum_{j\in N} v_{ij}$ and partition the interval $[0, \bar{v}_i]$ into L intervals $\{[\nu_i^{q-1}, \nu_i^q] : q = 1, \ldots, L\}$ with $0 = \nu_i^0 \leq \nu_i^1 \leq \ldots \leq \nu_i^{L-1} \leq \nu_i^L = \bar{v}_i$. Using the decision variables $x_i = \{x_{ij} : j \in N\} \in [0, 1]^n$, we define $\phi_i^{kq}(z)$ as

$$\phi_i^{kq}(z) \quad = \quad \max \quad (\nu_i^{q-1})^{\gamma_i} \left\{ \frac{\sum_{j\in N} v_{ij}\, r_{ij}\, x_{ij}}{\nu_i^{q-1}} - z \right\} \tag{21}$$

$$\text{st} \quad \sum_{j\in N} w_{ij}\, x_{ij} \leq b_i^k \tag{22}$$

$$\sum_{j\in N} v_{ij}\, x_{ij} \leq \nu_i^q \tag{23}$$

$$0 \leq x_{ij} \leq \mathbf{1}(w_{ij} \leq b_i^k) \qquad \forall\, j \in N, \tag{24}$$

which is a continuous knapsack problem with two dimensions. The selection of the intervals $\{[b_i^{k-1}, b_i^k] : k = 1, \ldots, K\}$ and $\{[\nu_i^{q-1}, \nu_i^q] : q = 1, \ldots, L\}$ can be completely arbitrary, as long as these intervals respectively cover $[0, c]$ and $[0, \bar{v}_i]$. We observe that $\phi_i^{kq}(z)$ is a linear function of z. If $q = 1$, then $\nu_i^{q-1} = 0$, in which case, we have a zero in the denominator of the fraction above. To deal with this case, if $q = 1$, then we follow the convention that $\phi_i^{kq}(z) = 0$ for all $k = 1, \ldots, K$ and $z \in \Re_+$. Roughly speaking, we can interpret problem (21)-(24) as a continuous version of problem (9). In the objective function of problem (21)-(24), the term ν_i^{q-1} corresponds to $V_i(S_i)$ in the objective function of problem (9). Noting that $R_i(S_i) = \sum_{j \in S_i} r_{ij}\, v_{ij} / V_i(S_i)$, the term $\sum_{j \in N} v_{ij}\, r_{ij}\, x_{ij} / \nu_i^{q-1}$ in the objective function of problem (21)-(24) corresponds to $R_i(S_i)$ in the objective function of problem (9). The first constraint in problem (21)-(24) imposes the capacity constraint, whereas the second constraint ensures that the total preference weight of the offered products are computed correctly. We use ν_i^{q-1} in the objective function, but ν_i^q in the constraint to ultimately ensure that we can use $\phi_i^{kq}(z)$ to obtain an upper bound on the optimal expected revenue. Using the decision variables Δ, $y = \{y_i : i \in M\}$ and z, to obtain an upper bound on the optimal expected revenue, we propose solving the problem

$$
\begin{aligned}
\min \quad & c\,\Delta + \sum_{i \in M} y_i && (25)\\
\text{st} \quad & b_i^{k-1}\,\Delta + y_i \geq \phi_i^{kq}(z) \qquad \forall\, i \in M,\ k = 1, \ldots, K,\ q = 1, \ldots, L && (26)\\
& c\,\Delta + \sum_{i \in M} y_i = v_0\, z && (27)\\
& \Delta \geq 0,\ y_i \text{ is free},\ z \text{ is free} \quad \forall\, i \in M. && (28)
\end{aligned}
$$

Since $\phi_i^{kq}(\cdot)$ is linear, the problem above is a linear program. The next theorem shows that we can use this problem to obtain an upper bound on the optimal expected revenue z^* in problem (1).

Theorem 7 *Letting $(\hat{\Delta}, \hat{y}, \hat{z})$ be an optimal solution to problem* (25)-(28)*, we have $\hat{z} \geq z^*$.*

Proof. We let $(S_1^*, \ldots, S_m^*)$ be an optimal solution to problem (1), k_i' be such that $C_i(S_i^*) \in [b_i^{k_i'-1}, b_i^{k_i'}]$ and q_i' be such that $V_i(S_i^*) \in [\nu_i^{q_i'-1}, \nu_i^{q_i'}]$. Since $(\hat{\Delta}, \hat{y}, \hat{z})$ is a feasible solution to problem (25)-(28), we have $b_i^{k_i'-1}\,\hat{\Delta} + \hat{y}_i \geq \phi_i^{k_i' q_i'}(\hat{z})$ for all $i \in M$. Adding this inequality over all $i \in M$, we obtain $\sum_{i \in M} \phi_i^{k_i' q_i'}(\hat{z}) \leq \sum_{i \in M} b_i^{k_i'-1}\,\hat{\Delta} + \sum_{i \in M} \hat{y}_i \leq \sum_{i \in M} C_i(S_i^*)\,\hat{\Delta} + \sum_{i \in M} \hat{y}_i \leq c\,\hat{\Delta} + \sum_{i \in M} \hat{y}_i = v_0\,\hat{z}$, where the second inequality uses the fact that $C_i(S_i^*) \geq b_i^{k_i'-1}$, the third inequality uses the fact that $(S_1^*, \ldots, S_m^*)$ is a feasible solution to problem (1) and the equality is by the fact that $(\hat{\Delta}, \hat{y}, \hat{z})$ is a feasible solution to problem (25)-(28). Thus, the last chain of inequalities implies that $\sum_{i \in M} \phi_i^{k_i' q_i'}(\hat{z}) \leq v_0\,\hat{z}$. On the other hand, consider a solution x_i^* to problem (21)-(24) obtained by letting $x_{ij}^* = 1$ if $j \in S_i^*$ and $x_{ij}^* = 0$ otherwise. Since $\sum_{j \in N} w_{ij}\, x_{ij}^* = C_i(S_i^*) \leq b_i^{k_i'}$ and $\sum_{j \in N} v_{ij}\, x_{ij}^* = V_i(S_i^*) \leq \nu_i^{q_i'}$, the solution x_i^* is feasible to problem (21)-(24) when we solve this problem with $k = k_i'$, $q = q_i'$ and $z = \hat{z}$. So, the optimal objective value of problem (21)-(24) is at least as large as the objective value provided by the feasible solution x_i^* and we obtain

$$\phi_i^{k_i' q_i'}(\hat z) \geq (\nu_i^{q_i'-1})^{\gamma_i} \left\{ \frac{\sum_{j \in N} v_{ij}\, r_{ij}\, x_{ij}^*}{\nu_i^{q_i'-1}} - \hat z \right\} = \frac{\sum_{j \in N} v_{ij}\, r_{ij}\, x_{ij}^*}{(\nu_i^{q_i'-1})^{1-\gamma_i}} - (\nu_i^{q_i'-1})^{\gamma_i}\, \hat z$$
$$\geq \frac{\sum_{j \in N} v_{ij}\, r_{ij}\, x_{ij}^*}{V_i(S_i^*)^{1-\gamma_i}} - V_i(S_i^*)^{\gamma_i}\, \hat z = V_i(S_i^*)^{\gamma_i} \left\{ \frac{\sum_{j \in S_i^*} v_{ij}\, r_{ij}}{V_i(S_i^*)} - \hat z \right\} = V_i(S_i^*)^{\gamma_i} (R_i(S_i^*) - \hat z),$$

where the second inequality uses the fact that $V_i(S_i^*) \geq \nu_i^{q_i'-1}$ and the second equality uses the definition of x_i^*. Since we have $\sum_{i \in M} \phi_i^{k_i' q_i'}(\hat z) \leq v_0 \hat z$ as shown at the beginning of the proof, the chain of inequalities above implies that $v_0\, \hat z \geq \sum_{i \in M} \phi_i^{k_i' q_i'}(\hat z) \geq \sum_{i \in M} V_i(S_i^*)^{\gamma_i}\, (R_i(S_i^*) - \hat z)$. If we focus on the first and last expressions in this chain of inequalities and solve for $\hat z$, then we obtain $\hat z \geq \sum_{i \in M} V_i(S_i^*)^{\gamma_i}\, R_i(S_i^*) / (v_0 + \sum_{i \in M} V_i(S_i^*)^{\gamma_i}) = \Pi(S_1^*, \ldots, S_m^*) = z^*$. $\square$