Online Appendix for
Bounding Optimal Expected Revenues for Assortment Optimization under Mixtures of Multinomial Logits

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A Online Appendix: Upper Bounds from a Linear Programming Formulation

Bront et al. (2009) show that problem (1) can be formulated as a mixed integer program. Thus, a tempting approach to obtain upper bounds on the optimal expected revenue is to solve the linear programming relaxation of this mixed integer program. In this section, we show that when we focus on each customer type individually, if customers of each type make a purchase with a reasonably large probability, then the optimal objective value of the linear programming relaxation of the mixed integer program is equal to the upper bound on the optimal expected revenue that is obtained under the assumption that we can offer different sets of products to different customer types. To state this result, we note that Bront et al. (2009) show that we can obtain the optimal objective value of problem (1) by solving the mixed integer program

$$
\begin{aligned}
\max \quad & \sum_{g \in G} \sum_{j \in N} \alpha^g \, r_j \, v_j^g \, y_j^g && && (18) \\
\text{subject to} \quad & \sum_{j \in N} v_j^g \, y_j^g + w^g = 1 && \forall \, g \in G \\
& y_j^g \le w^g && \forall \, j \in N, \; g \in G \\
& y_j^g \le z_j && \forall j \in N, \; g \in G \\
& w^g - y_j^g \le 1 - z_j && \forall j \in N, \; g \in G \\
& y_j^g \ge 0, \; w^g \ge 0, \; z_j \in \{0,1\} && \forall j \in N, \; g \in G.
\end{aligned}
$$

We use the vector $\hat{x}^g = \{\hat{x}_j^g : j \in N\} \in \{0,1\}^{|N|}$ to capture the set of products that maximizes the expected revenue only from customers of type g. In other words, $\hat{x}^g$ is an optimal solution to the problem $\hat{Z}^g = \max_{x \in \{0,1\}^{|N|}} \sum_{j \in N} r_j \, P_j^g(x)$ with the corresponding optimal objective value $\hat{Z}^g$. Therefore, the expected revenue $\sum_{g \in G} \alpha^g \, \hat{Z}^g$ provides an upper bound on the optimal objective value of problem (1), since the expected revenue $\sum_{g \in G} \alpha^g \, \hat{Z}^g$ is obtained under the assumption that we can offer different sets of products to different customer types, whereas problem (1) requires that we offer a single set of products to customers of all types. In this case, if we offer the set of products captured by the vector $\hat{x}^g$, then a customer of type g makes a purchase within the set of offered products with probability $\hat{P}^g = \sum_{j \in N} P_j^g(\hat{x}^g)$.

The next proposition shows that if we have $\hat{P}^g \ge 1/2$ for all $g \in G$, then the optimal objective value of the linear programming relaxation of problem (18) is equal to $\sum_{g \in G} \alpha^g \, \hat{Z}^g$. So, if customers of each type g make a purchase with a probability of at least 1/2 when offered the set of products captured by the vector $\hat{x}^g$, then the upper bound on the optimal expected revenue provided by the linear programming relaxation of problem (18) does not improve the upper bound obtained under the assumption that we can offer different sets of products to different customer types.

Proposition 4 *Let $\hat{x}^g$ be an optimal solution to the problem $\max_{x \in \{0,1\}^{|N|}} \sum_{j \in N} r_j \, P_j^g(x)$ with the corresponding optimal objective value $\hat{Z}^g$ and $\hat{P}^g$ be defined as $\hat{P}^g = \sum_{j \in N} P_j^g(\hat{x}^g)$. If we have $\hat{P}^g \ge 1/2$ for all $g \in G$, then the optimal objective value of the linear programming relaxation of problem* (18) *is equal to $\sum_{g \in G} \alpha^g \, \hat{Z}^g$.*

Proof. We let $\hat{\zeta}$ be the optimal objective value of the linear programming relaxation of problem (18). First, we show that $\hat{\zeta} \leq \sum_{g \in G} \alpha^g \hat{Z}^g$. We use $\hat{y} = \{\hat{y}^g_j : j \in N,\ g \in G\}$, $\hat{w} = \{\hat{w}^g : g \in G\}$ and $\hat{z} = \{\hat{z}_j : j \in N\}$ to denote an optimal solution to the linear programming relaxation of problem (18) and let $\hat{\zeta}^g$ be defined as $\hat{\zeta}^g = \sum_{j \in N} r_j\, v^g_j\, \hat{y}^g_j$, in which case, we have $\hat{\zeta} = \sum_{g \in G} \alpha^g\, \hat{\zeta}^g$. We observe that we must have $\hat{w}^g > 0$ for all $g \in G$, since if $\hat{w}^g = 0$ for some $g \in G$, then the second set of constraints in problem (18) imply that $\hat{y}^g_j = 0$ for all $j \in N$ as well, in which case, it is not possible to satisfy the first set of constraints. Now, we claim that $\hat{\zeta}^g \leq \hat{Z}^g$. To get a contradiction, we proceed under the assumption that $\hat{\zeta}^g > \hat{Z}^g$. By the definition of $\hat{Z}^g$, we have $\hat{Z}^g \geq \sum_{j \in N} r_j\, P^g_j(x) = (\sum_{j \in N} r_j\, v^g_j\, x^g_j)/(1 + \sum_{j \in N} v^g_j\, x^g_j)$ for all $x^g \in \{0,1\}^{|N|}$. If we arrange the terms in this inequality, then it follows that $\hat{Z}^g \geq \sum_{j \in N} (r_j - \hat{Z}^g)\, v^g_j\, x^g_j$ for all $x^g \in \{0,1\}^{|N|}$, in which case, we obtain the chain of inequalities

$$\zeta^g > \hat{Z}^g \geq \max_{x^g \in \{0,1\}^{|N|}} \left\{ \sum_{j \in N} (r_j - \hat{Z}^g)\, v^g_j\, x^g_j \right\} \geq \max_{x^g \in \{0,1\}^{|N|}} \left\{ \sum_{j \in N} (r_j - \hat{\zeta}^g)\, v^g_j\, x^g_j \right\}$$
$$= \max_{x^g \in [0,1]^{|N|}} \left\{ \sum_{j \in N} (r_j - \hat{\zeta}^g)\, v^g_j\, x^g_j \right\} \geq \sum_{j \in N} (r_j - \hat{\zeta}^g)\, v^g_j\, \frac{\hat{y}^g_j}{\hat{w}^g}.$$

The first and third inequalities above use the assumption that $\hat{\zeta}^g > \hat{Z}^g$. The equality above follows by noting that the objective function of the second optimization problem above is linear, in which case, the continuous relaxation of this problem has an integer optimal solution. To see that the fourth inequality above holds, we note that since $\hat{w}^g > 0$, the second set of constraints in problem (18) yield $\hat{y}^g_j / \hat{w}^g \in [0,1]$ for all $j \in N$ so that $\{\hat{y}^g_j / \hat{w}^g : j \in N\}$ is a feasible solution to the third optimization problem above. From the chain of inequalities above, we obtain $\sum_{j \in N} (r_j - \hat{\zeta}^g)\, v^g_j\, \hat{y}^g_j / \hat{w}^g < \hat{\zeta}^g$, which can equivalently be written as $\sum_{j \in N} r_j\, v^g_j\, \hat{y}^g_j < \hat{\zeta}^g\, (\hat{w}^g + \sum_{j \in N} v^g_j\, \hat{y}^g_j)$. By the definition of $\hat{\zeta}^g$, the left side of the last strict inequality is equal to $\hat{\zeta}^g$, but noting the first set of constraints in problem (18), we have $\hat{w}^g + \sum_{j \in N} v^g_j\, \hat{y}^g_j = 1$ and the right of this strict inequality is equal to $\hat{\zeta}^g$ as well, which is a contradiction. Thus, our claim holds and we have $\hat{\zeta}^g \leq \hat{Z}^g$. In this case, we obtain $\hat{\zeta} = \sum_{g \in G} \alpha^g\, \hat{\zeta}^g \leq \sum_{g \in G} \alpha^g\, \hat{Z}^g$.

Second, we show that $\hat{\zeta} \geq \sum_{g \in G} \alpha^g\, \hat{Z}^g$. Letting $\hat{x}^g = \{\hat{x}^g_j : j \in N\}$ be defined as in the statement of the proposition, we define the solution $\hat{y} = \{\hat{y}^g_j : j \in N,\ g \in G\}$, $\hat{w} = \{\hat{w}^g : g \in G\}$ and $\hat{z} = \{\hat{z}_j : j \in N\}$ to the linear programming relaxation of problem (18) as

$$\hat{y}^g_j = \frac{\hat{x}^g_j}{1 + \sum_{i \in N} v^g_i\, \hat{x}^g_i} \qquad \hat{w}^g = \frac{1}{1 + \sum_{j \in N} v^g_j\, \hat{x}^g_j} \qquad \hat{z}_j = \max_{g \in G} \left\{ \frac{\hat{x}^g_j}{1 + \sum_{i \in N} v^g_i\, \hat{x}^g_i} \right\}.$$

It is straightforward to see that the solution $(\hat{y}, \hat{w}, \hat{z})$ satisfies the first, second and third sets of constraints in problem (18). Since $\hat{P}^g = (\sum_{j \in N} v^g_j\, \hat{x}^g_j)/(1 + \sum_{j \in N} v^g_j\, \hat{x}^g_j) \geq 1/2$, subtracting one from both sides of this inequality, we obtain $1/(1+\sum_{j \in N} v^g_j\, \hat{x}^g_j) \leq 1/2$, which implies that $\hat{y}^g_j \leq 1/2$, $\hat{w}^g \leq 1/2$ and $\hat{z}_j \leq 1/2$ for all $j \in N$, $g \in G$. Also, by the definition of $\hat{y}^g_j$ and $\hat{w}^g$, we have either $\hat{w}^g - \hat{y}^g_j = 0$ or $\hat{w}^g - \hat{y}^g_j = \hat{w}^g$, which happen respectively when $\hat{x}^g_j = 1$ and $\hat{x}^g_j = 0$. If we have $\hat{w}^g - \hat{y}^g_j = 0$, then the fourth set of constraints for this product j and customer type g is satisfied. If we have $\hat{w}^g - \hat{y}^g_j = \hat{w}^g$, then we obtain $\hat{w}^g - \hat{y}^g_j = \hat{w}^g \leq 1/2 = 1 - 1/2 \leq 1 - \hat{z}_j$, indicating that the fourth set of constraints for this product j and customer type g is satisfied as well. Thus, the solution $(\hat{y}, \hat{w}, \hat{z})$ is feasible to the linear programming relaxation of problem (18). So, the optimal objective value of the linear programming relaxation of problem (18) satisfies $\hat{\zeta} \geq \sum_{g \in G} \sum_{j \in N} \alpha^g \, r_j \, v^g_j \, \hat{y}^g_j = \sum_{g \in G} \alpha^g \sum_{j \in N} r_j \, v^g_j \, \hat{x}^g_j / (1 + \sum_{i \in N} v^g_i \, \hat{x}^g_i) = \sum_{g \in G} \sum_{j \in N} \alpha^g \, r_j \, P^g_j(\hat{x}^g) = \sum_{g \in G} \alpha^g \, \hat{Z}^g$, where the first inequality follows from the fact that $(\hat{y}, \hat{w}, \hat{z})$ is a feasible, but not necessarily an optimal solution to the linear programming relaxation of problem (18). $\square$

The first part of the proof of Proposition 4 does not use the assumption that $\hat{P}^g \geq 1/2$ for all $g \in G$. Therefore, the upper bound on the optimal expected revenue provided by the linear programming relaxation of problem (18) is always at least as tight as the upper bound that is obtained under the assumption that we can offer different sets of products to different customer types. However, when we focus on each customer type individually, if customers of each type make a purchase with a probability exceeding 1/2, then the upper bound provided by the linear programming relaxation of problem (18) is equal to the upper bound that is obtained under the assumption that we can offer different sets of products to different customer types.

If we do not have $\hat{P}^g \geq 1/2$ for all $g \in G$, then we can give examples where the upper bound provided the linear programming relaxation of problem (18) is tighter than the upper bound obtained under the assumption that we can offer different sets of products to different customer types. Consider a problem instance with two products $N = \{1, 2\}$ and two customer types $G = \{1, 2\}$. The revenues of the products are $(r_1, r_2) = (95, 7)$. The preference weights of the two customer types are $(v^1_1, v^1_2) = (0.09, 0.09)$ and $(v^2_1, v^2_2) = (0, 0.01)$. The probabilities of observing the two customer types are $(\alpha^1, \alpha^2) = (0.5, 0.5)$. For this problem instance, the optimal objective value of the linear programming relaxation of problem (18) is about 3.92. On the other hand, if we assume that we can offer different sets of products to different customer types, then the upper bound that we obtain is about 3.96. For this problem instance, the solutions $\hat{x}^1 = (1, 0)$ and $\hat{x}^2 = (1, 1)$ maximize the expected revenue from each one of the two customer types when we focus on each one of the two customer types individually. When customers of each type are offered the solutions corresponding to them, they make a purchase respectively with probabilities $0.09/(1 + 0.09) \approx 0.08$ and $0.01/(1 + 0.01) \approx 0.01$. Since these probabilities of making a purchase are less than 1/2, this example violates the assumption of Proposition 4 and the upper bound on the optimal expected revenue provided by the linear programming relaxation of problem (18) can be tighter than the upper bound that is obtained under the assumption that we can offer different sets of products to different customer types.

B Online Appendix: Expected Revenue for Specially Structured Problem Instances

In this section, we show that the optimal expected revenue in the specially structured problem instance given in Table 3 is at most $(3 + 2\,\theta + \theta^2)/(1 + \theta + \theta^2)$. We observe that the revenue

Offered Set of Prods.	Expected Revenue from Cus. Typ. 1	2	3	Upp. Bnd. on Exp. Rev.
{3}	$\frac{\theta^4}{1+\theta^2}$	0	0	$\frac{\theta^2}{1+\theta+\theta^2}$
{1,3}	$\frac{\theta^4+\theta^6}{1+\theta^2+\theta^6}$	$\frac{\theta^6}{1+\theta^6}$	$\frac{\theta^6}{1+\theta^6}$	$\frac{2+\theta+\theta^2}{1+\theta+\theta^2}$
{2,3}	$\frac{\theta^4+\theta^5}{1+\theta^2+\theta^4}$	$\frac{\theta^5}{1+\theta^4}$	0	$\frac{2\,\theta+\theta^2}{1+\theta+\theta^2}$
{1,2,3}	$\frac{\theta^4+\theta^5+\theta^6}{1+\theta^2+\theta^4+\theta^6}$	$\frac{\theta^5+\theta^6}{1+\theta^4+\theta^6}$	$\frac{\theta^6}{1+\theta^6}$	$\frac{3+2\,\theta+\theta^2}{1+\theta+\theta^2}$

Table 8: Expected revenues obtained from different customer types and an upper bound on expected revenues obtained by offering different sets of products.

associated with product three is θ^2, which is the largest one among all product revenues. Thus, it is trivially optimal to offer product three and we only consider the sets of products that include product three without loss of optimality. Table 8 shows the expected revenue obtained from each customer type when we offer a particular set of products and an upper bound on the expected revenue over all customer types when we offer the particular set. To understand the figures in Table 8, for example, if we offer the set of products $\{2,3\}$, then we obtain expected revenues of $\sum_{j\in\{2,3\}} r_j\, v_j^1/(1+\sum_{j\in\{2,3\}} v_j^1) = (\theta^4+\theta^5)/(1+\theta^2+\theta^4)$, $\sum_{j\in\{2,3\}} r_j\, v_j^2/(1+\sum_{j\in\{2,3\}} v_j^2) = \theta^5/(1+\theta^4)$ and zero respectively from customers of type one, two and three. These figures correspond to the entries in the second, third and fourth columns in Table 8. Noting that $\theta \geq 1$, we have $(\theta^4+\theta^5)/(1+\theta^2+\theta^4) \leq 2\,\theta^5/\theta^4 = 2\,\theta$ and $\theta^5/(1+\theta^4) \leq \theta^5/\theta^4 = \theta$, in which case, if we offer the set of products $\{2,3\}$, then the expected revenue obtained over all customer types is at most

$$\frac{1}{1+\theta+\theta^2}\,2\,\theta + \frac{\theta}{1+\theta+\theta^2}\,\theta = \frac{2\,\theta+\theta^2}{1+\theta+\theta^2},$$

which corresponds to the entry in the last column in Table 8. The other entries in Table 8 are obtained in a similar fashion. We observe that each one of the entries in the last column of Table 8 is no larger than $(3+2\,\theta+\theta^2)/(1+\theta+\theta^2)$, which implies that the optimal expected revenue for the problem instance given in Table 3 is at most $(3+2\,\theta+\theta^2)/(1+\theta+\theta^2)$, as desired.

References

Bront, J. J. M., Diaz, I. M. and Vulcano, G. (2009), 'A column generation algorithm for choice-based network revenue management', *Operations Research* **57**(3), 769–784.